Sketch of solution to Homework 1 Part B

Q9 Let $f, g \in R[0,1]$ with $f \leq g$. Take a partition $\mathcal{P}$, we have

$$\sum_{i=1}^{n} M_i(f)\Delta x_i \leq \sum_{i=1}^{n} M_i(g)\Delta x_i$$

where $M_i(f) = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$. Since f and g are Riemann integrable, we therefore conclude that

$$\int_0^1 f dx \leq \sum_{i=1}^{n} M_i(f)\Delta x_i \leq \sum_{i=1}^{n} M_i(g)\Delta x_i.$$

Taking inf on Right hand side, we conclude that $\int_0^1 f \leq \int_0^1 g$. Linearity follows similarly.

Suppose f_n converges to f uniformly, then for any $\epsilon > 0$, there is N such that for all $x \in [0,1]$, $n > N$,

$$|f(x) - f_n(x)| < \epsilon.$$

Then for all $n > N$,

$$\left| \int_0^1 f_n - \int_0^1 f \right| \leq \int_0^1 |f(x) - f_n(x)| dx \leq \epsilon.$$